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FOLIATIONS AND FIBRATIONS

CLIFFORD J. EARLE & JAMES EELLS, JR.

1. Introduction

Let X and Y be differentiable manifolds modeled on Banach spaces, and $f: X \to Y$ a differentiable surjective map. Suppose that for each point $x \varepsilon X$ the differential $f_*(x)$ of f at x maps the tangent space $X(x)$ to X at x surjectively onto that $Y(f(x))$ to Y at $f(x)$. If the kernel of $f_*(x)$ is a direct summand of $X(x)$ (i.e., Ker $f_*(x)$ is a closed linear subspace of $X(x)$ admitting a closed supplement), then the sets $(f^{-1}(y))_{y \varepsilon Y}$ are closed differentiable submanifolds of X defining a foliation, whose leaves are the components of the manifolds $f^{-1}(y)$. We will say briefly that f *foliates* X. In particular, with each $x \varepsilon X$ there are neighborhoods U_x, $V_{f(x)}$ of x, $f(x)$ such that $f \mid U_x \to V_{f(x)}$ is a trivial fibration; see [4], [8].

The object of this paper is to impose natural conditions sufficient to insure that such a map $f: X \to Y$ is a locally trivial fibration. In finite dimensions there are several instances in which such conditions have been displayed, based on methods of Riemannian geometry; see § 4 below. In particular, the work of Ehresmann [5] can be viewed as our starting point. Theorem 3C below (which was announced in [4, §5G]) includes these cases, and provides a suitable general criterion, avoiding hypotheses of finite dimensionality, separability, or of special Banach space structure of the models. Its proof therefore is necessarily different from those of the finite dimensional cases; it is based on properties of ordinary differential equations in Banach spaces, used to construct coherent liftings of paths.

In § 4 we produce several special cases of Theorem 3C, extending theorems of Ehresmann [5], Hermann [6], and Rinehart [11]. Moreover, there is an important application of our theorem in the infinite dimensional case to the Teichmüller theory of Fuchsian groups. That is the object of our study [3].

2. Local lifting

In this section we adapt standard theory [2] of ordinary differential equations in Banach spaces to obtain a local path lifting property. We defer further geometric interpretation until the following section.

(A) Let E and F be Banach spaces with norms $|\ |_E$ and $|\ |_F$, and $L(F, E)$ the Banach space of all continuous linear maps of F into E, with norm denot-

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ed by $\|\ \|$. If U is an open subset of E, we consider a map $\sigma : U \to L(F, E)$ satisfying a local Lipschitz condition on U; i.e., for every point $x_0 \varepsilon U$ there is a neighborhood U_0 of x_0 contained in U and a number K_0 such that

$$\|\sigma(x_1) - \sigma(x_2)\| \leq K_0 |x_1 - x_2|_E \quad \text{for all } x_1, x_2 \varepsilon U_0.$$

Let $C^1(I, F)$ denote the Banach space of maps $v : I \to F$ of class C^1, with norm

$$|v|_{C^1(I,F)} = \sup\{|v(t)|_F : t\varepsilon I\} + \sup\{|v'(t)|_F : t\varepsilon I\}.$$

Here I denotes the closed unit interval. Note that σ induces a map (still called σ)

$$\sigma : U \to L[C^1(I, F), C^1(I, E)]$$

through the formula $(\sigma(x)v)t = \sigma(x)v(t)$ for all $t\varepsilon I$. It is clear that each $\sigma(x)v \varepsilon C^1(I, E)$, being the composition of two C^1-maps. Furthermore, each $\sigma(x)$ is linear and continuous; for

$$|(\sigma(x)v)t|_E \leq \|\sigma(x)\| \, |v(t)|_F, \quad |(\sigma(x)v)'t|_E \leq \|\sigma(x)\| \, |v'(t)|_F.$$

Finally, the induced map is locally Lipschitz on U, a property which follows at once from the local Lipschitz character of $\sigma : U \to L(F, E)$ and the estimate

$$|(\sigma(x_1) - \sigma(x_2))v|_{C^1(I,E)} \leq \|\sigma(x_1) - \sigma(x_2)\| \, |v|_{C^1(I,F)}.$$

(B) Let us set $H = U \times C^1(I, F)$, and define $\tilde{\sigma} : I \times H \to E$ by $\tilde{\sigma}(t; x, v) = \sigma(x)v(t)$. Then $\tilde{\sigma}$ is continuous and is locally Lipschitz on H, uniformly on I; that follows from the estimate (taking $t_1 = t$)

$$|\tilde{\sigma}(t_1; x_1, v_1) - \tilde{\sigma}(t; x, v)|_E \leq \|\sigma(x_1)\| \, |v_1(t_1) - v(t)|_F + \|\sigma(x_1) - \sigma(x)\| \, |v(t)|_F.$$

Lemma. *Given a point $(x_0, v_0) \varepsilon H$, there is a closed non-trivial interval $I_0 = [0, t_0]$ and a unique C^1-map $t \to h(t) = h(t; x_0, v_0)$ defined on I_0 with values in U and satisfying the differential equation*

$$\begin{aligned} h'(t; x_0, v_0) &= \tilde{\sigma}(t; h(t; x_0, v_0), v_0) \\ h(0; x_0, v_0) &= x_0 . \end{aligned} \tag{1}$$

We have formulated this statement for a closed interval I_0 for minor technical convenience; the basic existence theorem quoted below assures the existence of an open 0-centered interval on which h is defined and unique.

Proof. Define $f_{v_0} : I \times H \to E \times C^1(I, F)$ by $f_{v_0}(t; (x, v)) = (\tilde{\sigma}(t; x, v_0), 0)$. Then f_{v_0} is continuous, and is locally Lipschitz on H, uniformly on I. It follows from [2, §§10.4.5-10.4.6] that there is a non-trivial interval $I_0 = [0, t_0]$

contained in an open interval and a unique C^1-map $u: I_0 \to H$ such that $u'(t) = f_{r_0}(t, u(t))$ and $u(0) = (x, v_0)$. If we write $u(t) = (h(t), g(t))$, then $g(t) \equiv v_0$ and $h'(t) = \tilde{\sigma}(t; h(t), v_0)$, $h(0) = x$. Therefore $h: I_0 \to U$ is the desired solution of (1).

Lemma. *For any point $(x_0, v_0) \varepsilon H$ there is a neighborhood $H_0 \subset H$ and an open interval $I_0 \subset I$ such that (1) has a unique solution h defined on I_0 with initial conditions $(x, v) \varepsilon H_0$. Furthermore, the maps $h: I_0 \times H_0 \to U$ and $h': I_0 \times H_0 \to E$ are continuous.*

This is a simple adaptation of [2, §10.8.1].

Lemma. *Fix any $(x_1, v_1) \varepsilon H_0$ and $\varepsilon > 0$. Then there is a neighborhood $H_1 \subset H_0$ of (x_1, v_1) such that*

$$|h_2 - h_1|_{C^1(I_0, E)} < \varepsilon \quad \textit{for all } (x_2, v_2) \varepsilon H_1,$$

where $h_k(t) = h(t; x_k, v_k)$.

Proof. Since h_1 is uniformly continuous on the closed interval I_0, there is a number $\delta > 0$ such that $|h_1(s) - h_1(t)| < \varepsilon / 4$ whenever $|s - t| < \delta$. For each $t \varepsilon I_0$ the preceding lemma assures the existence of a neighborhood $H(t)$ of (x_1, v_1) and an interval $I(t)$ of the form $I(t) = \{ s \varepsilon I_0 : |s - t| < \delta_t \}$ for some $\delta_t > 0$, such that $|h_2(s) - h_1(t)| < \varepsilon / 4$ if $(x_2, v_2) \varepsilon H(t)$ and $s \varepsilon I(t)$. From the compactness of I_0 we select a finite number of points $(t_i)_{1 \le i \le m}$ for which $(I(t_i))_{1 \le i \le m}$ cover I_0. Then for any $t \varepsilon I_0$ we can find some $t_i \varepsilon I_0$ for which $|t - t_i| < \delta_{t_i} < \delta$. Set $H_1 = H(t_1) \cap \ldots \cap H(t_m)$, a neighborhood of (x_1, v_1). Then if $(x_2, v_2) \varepsilon H_1 \subset H(t_i)$, we have

$$|h_2(t) - h_1(t)| \le |h_2(t) - h_1(t_i)| + |h_1(t_i) - h_1(t)|,$$

and the first term in the right member is $< \varepsilon / 4$; the second term is $< \varepsilon / 4$ by our choice of δ. Thus $|h_2 - h_1|_{C^0(I_0, E)} < \varepsilon / 2$. That we can also choose H_1 so that $|h_2' - h_1'|_{C^0(I_0, E)} < \varepsilon / 2$ follows from the local Lipschitz character of $\tilde{\sigma}$.

(C) We consolidate our position in the

Proposition. *Let E and F be Banach spaces, and $U \subset E$ an open set. Given a locally Lipschitz map $\sigma: U \to L(F, E)$, we define*

$$\tilde{\sigma}: I \times U \times C^1(I, F) \to E \textit{ by } \tilde{\sigma}(t; x, v) = \sigma(x) v(t).$$

Then for each $(x_0, v_0) \varepsilon U \times C^1(I, F)$ there is a closed interval $I_0 = [0, t_0]$ and neighborhoods $U_0 \subset U$ and $P_0 \subset C^1(I, F)$ of x_0 and v_0 on which (1) has a unique solution $h: I_0 \times U_0 \times P_0 \to U$. Moreover, the induced map $\bar{h}: U_0 \times P_0 \to C^1(I_0, U)$ defined by $(\bar{h}(x, v))t = h(t; x, v)$ is continuous.

3. Fibrations of Finsler manifolds

(A) Let X be a paracompact C^1-manifold modeled on a Banach space E; it is well known that the paracompactness condition on X is equivalent to its metrizability, in virtue of theorems of A. Stone and Y. Smirnov. Let $\pi : T(X) \to X$ denote the tangent vector bundle of X, and $X(x) = \pi^{-1}(x)$ the tangent space to X at x. Thus each $X(x)$ is a locally convex topological vector space admitting a compatible Banach space structure; and $T(X)$ is a paracompact C^0-manifold modeled on a Banach space.

Definition. A *Finsler structure on* X is a continuous assignment of a norm α_x to each tangent space $X(x)$ which is compatible with its Banach space structure, such that each point of X is centered at a coordinate chart (θ, U) on X for which there are numbers $A, B > 0$ such that

$$A\alpha_x(v) \leq |\theta_*(x)v|_E \leq B\alpha_x(v) \tag{2}$$

for all $x \varepsilon U$, $v \varepsilon X(x)$. Here $\theta_*(x)$ denotes the differential of θ at x.

Since X is paracompact, there are continuous partitions of unity subordinate to any open cover of X. It therefore follows from standard reasoning that every X admits a Finsler structure. Furthermore, if we introduce a metric on X we can suppose that α is locally Lipschitz.

For any C^1-path $b : [t_0, t_1] \to X$ we define its *length*

$$L(b) = L_{t_0}^{t_1}(b) = \int_{t_0}^{t_1} \alpha_{b(t)}(b'(t))dt\,.$$

Similarly, if the domain of b is a half-open interval $[t_0, t_1]$, we define

$$L(b) = \lim_{t \to t_1} L_{t_0}^{t}(b), \text{ where } 0 \leq t_0 \leq t < t_1\,.$$

Assuming that X is connected we define the *Finsler distance* $\sigma(x_0, x_1)$ between two points $x_0, x_1 \varepsilon X$ by $\sigma(x_0, x_1) = \inf\{L(b) : b$ is a piecewise C^1-path on X joining x_0 to $x_1\}$. The following lemma is proved as in the finite dimensional case [10]:

Lemma. *If* (X, α) *is a Finsler manifold, then the Finsler distance is a metric on* X *compatible with its topology.*

In the future, when speaking of a *complete Finsler manifold* (X, α) we refer to completeness with respect to Cauchy sequences on X relative to the metric σ.

(B) If (X, α) and (Y, β) are Finsler manifolds and $f : X \to Y$ is a C^1-map foliating X as in the Introduction, then the tangent bundle $T(X)$ admits a C^0-direct sum decomposition; in fact, if $f^{-1}T(Y) \to X$ denotes the vector bundle over X obtained by pulling back $T(Y)$ via f, and K is the subbundle of $T(X)$ whose fibre over x is $K_x = Ker\, f_*(x)$, then the exact sequence

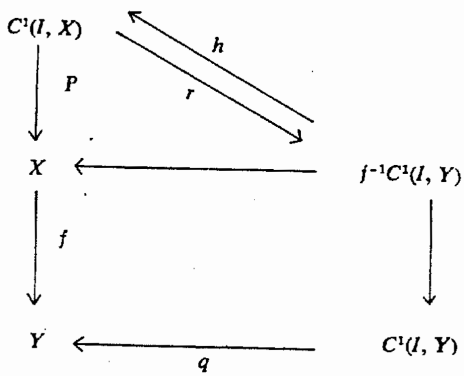
$$0 \to K \to T(X) \xrightarrow{f_*} f^{-1}T(Y) \to 0 \tag{3}$$

admits locally Lipschitz splittings; i.e., bundle maps $s : f^{-1}T(Y) \to T(X)$ which are locally Lipschitz, are continuous and linear in each fibre, and such that $f_*(x)s(x)$ is the identity map on $Y(f(x))$ for all $x \varepsilon X$. Again, that is a matter of using a locally Lipschitz partition of unity subordinate to a given locally finite open cover of X. Let us say that a path $b\varepsilon C^1(I, X)$ *is horizontal relative to* s if every tangent vector $b'(t)$ belongs to the image space $s(f(b(t)))Y(f(b(t)))$.

Our aim now is to use the splitting s to construct a local horizontal lifting of C^1-paths, and to do so in a coherent manner. Let $p: C^1(I, X) \to X$ and $q: C^1(I, Y) \to Y$ be the Serre maps $p(b) = b(0)$ and $q(c) = c(0)$, respectively. Using the notation of induced fibrations, let

$$f^{-1}C^1(I, Y) = \{(x, c)\varepsilon X \times C^1(I, Y): f(x) = q(c)\}.$$

In the spirit of Hurewicz's definition [7] of fibre space in the topological context, let us define a *Hurewicz C^1-connection for $f: X \to Y$* to be a continuous section of the map $r: C^1(I, X) \to f^{-1}C^1(I, Y)$ given by $r(b) = (b(0), f\circ b)$; i.e., a map h such that $r^0 h =$ identity:



We can now apply Proposition 2C as follows: Given $(x, c)\varepsilon f^{-1}C^1(I, Y)$ there is a subinterval $I_0 = [0, t_0] \subset I$, a neighborhood of the form $W_0 = (U_0 \times P_0) \cap f^{-1}C^1(I, Y)$, and a continuous map $h: W_0 \to C^1(I_0, U_0)$, where 1) $f\circ h(c) = c$ on I_0 and 2) $h'(x,c)$ is horizontal. Thus a locally Lipschitz splitting s defines a sort of local Hurewicz connection for $f: X \to Y$.

Lemma. *Suppose that (X, α) is complete. Given any $(x_0, c)\varepsilon f^{-1}C^1(I, Y)$, define $b(t) = h(t; x_0, c)$ in its maximal domain $I_0 = [0, \bar{t})$ in I. If its length*

$L_0^{\bar{t}}(b) < \infty$, then $\bar{t} = 1$.

Puoof. Suppose $\bar{t} < 1$. Then b can be extended to a continuous map on $\bar{I}_0$: For $(b(t))_{0 \leq t < \bar{t}}$ is totally bounded in X, because we can partition I_0 into subintervals (t_i, t_{i+1}) over which each $\int_{t_i}^{t_{i+1}} \alpha(b'(\tau))d\tau$ is arbitrarily small. Since X is complete, $(b(t))$ converges to some point $\bar{x}\varepsilon X$ as $t \to \bar{t}$; note also that $f(\bar{x}) = c(\bar{t})$. We define $b(\bar{t}) = \bar{x}$, whence b is continuous on $\bar{I}_0$.

We next show that $b \varepsilon C^1(\bar{I}_0, X)$ by establishing that we can define $b'(\bar{t}) = \lim b'(t_i)$, and that $b'(\bar{t})$ is horizontal. We work in a coordinate chart:

$$|b'(t_i) - b'(t_j)| = |s(b(t_i))c'(t_i) - s(b(t_j))c'(t_j)|$$
$$\leq \| s(b(t_i)) \| \, |c'(t_i) - c'(t_j)| + \| s(b(t_i)) - s(b(t_j)) \| \, |c'(t_j)| \, .$$

Since $s \circ b$ is locally Lipschitz and $(c'(t_i))_{i \geq 1}$ is Cauchy, it follows that $(b'(t_i))_{i \geq 1}$ is Cauchy. But $b'(t_i) = s(b(t_i))c'(t_i) \to s(b(\bar{t}))c'(\bar{t})$, so that $b'(t_i) \to b'(\bar{t})$ as $t_i \to \bar{t}$.

By considering our local Hurewicz connection at $(\bar{x}, c)$ we find that we can extend the domain of b in a C^1-manner to an open interval in I containing $\bar{I}_0$, thus contradicting the maximality property of $\bar{t}$.

(C) Lemma 3B suggests conditions on the splitting sufficient to insure that h define a Hurewicz connection for $f: X \to Y$; e.g., that $x \to \| s(x) \|_x$ be uniformly bounded, where the norm on $L(Y(f(x)), X(x))$ is defined using the Finsler structures $\beta_{f(x)}$ and α_x. Let us say that a locally Lipschitz splitting s *is bounded locally over* Y if for each $y_0 \varepsilon Y$ there is a number $\eta_0 > 0$ and a neighborhood V_0 of y_0 such that $\| s(x) \|_x \leq \eta_0$ for all $x \varepsilon f^{-1}(V_0)$.

Remark. If s is bounded on the fibres of X, then s is bounded locally over an open everywhere dense subset of Y, by the Baire category theorem. Furthermore, if the fibres of X are compact, then s is bounded locally over Y.

The following result is our main objective.

Theorem. *Let (X, α), (Y, β) be Finsler C^1-manifolds modeled on Banach spaces, and suppose that (X, α) is complete. Let $f: X \to Y$ be a surjective C^1-map which foliates X. If there is a locally Lipschitz splitting of the sequence (3) which is bounded locally over Y, then $f: X \to Y$ is a locally C^0-trivial fibration.*

Proof. First of all, our boundedness hypothesis and Lemma 3B imply that every horizontal lift $t \to b(t) = h(t; x_0, c)$ is defined for all $t \varepsilon I$, since $|b'(t)| \leq \| s(b(t)) \| \, |c'(t)|$; therefore h is a Hurewicz connection. Next, we take $y_0 \varepsilon Y$ and a coordinate chart (θ_0, V_0) centered at y_0 over which s is bounded by $\eta_0 > 0$. Thus for every $y \varepsilon V_0$ we have a unique straight (relative to θ_0) line segment μ_y joining y to y_0. Setting $X_0 = f^{-1}(y_0)$, we define $\phi_0 : f^{-1}(V_0) \to V_0 \times X_0$ as follows: For any $x \varepsilon f^{-1}(V_0)$ we take the path $\mu_{f(x)}$ in V_0, and construct the unique horizontal lift λ_x starting at x and ending in X_0. Set $\phi_0(x) = (f(x), \lambda_x(1))$. The map ϕ_0 is bijective; for if $(y, z) \varepsilon V_0 \times X_0$, there is a unique path μ_y and lift λ ending at z, whence $\phi_0(\lambda(0)) = (y, z)$. The bicontinuity of ϕ_0 follows from the continuity of solutions of (1) with respect to initial conditions.

Remarks. There are analogous theorems asserting that a C^k-map f foliating X defines a locally C^k-trivial fibration; we should assume that the manifolds admit C^k-partitions of unity. Furthermore, in special Riemannian contexts in finite dimensions (e.g., when the fibres are totally geodesic) it is known [6] that f defines a differentiable fibre bundle with Lie structural group. We shall not pursue generalizations of these properties at this time.

(D) If $f: X \to Y$ is a map satisfying the hypotheses of Theorem 3C, then we have the exact homotopy sequence

$$\cdots \to \pi_i(X) \xrightarrow{f_i} \pi_i(Y) \xrightarrow{\partial_i} \pi_{i-1}(X_0) \to \pi_{i-1}(X) \to \cdots$$
$$\xrightarrow{f_1} \pi_1(Y) \to \pi_0(X_0) \to \pi_0(X) .$$

It is a general property of paracompact manifolds modeled on Banach spaces that they are absolute neighborhood retracts; in particular, a manifold is contractible when and only when it is an absolute retract. On the other hand, a theorem of J.H.C. Whitehead asserts that in the class of absolute neighborhood retracts a map $f: X \to Y$ is a homotopy equivalence if and only if f induces isomorphisms $f_i : \pi_i(X) \to \pi_i(Y)$ of the homotopy groups for all $i \varepsilon Z$; in particular, X is contractible if and only if all $\pi_i(X) = 0$. (Relative to these assertions see [4, § 4] and [9].) We obtain the

Corollary. *Suppose that X is connected. Then with the hypotheses of Theorem 3C, the fibre X_0 is contractible if and only if $f: X \to Y$ is a homotopy equivalence.*

4. Special cases

(A) First of all, let us extend a theorem of Ehresmann [5].

Recall that a map $f: X \to Y$ is *proper* if the inverse image of every compact set is compact. In our situation that is equivalent to saying that the fibre $f^{-1}(y_0)$ is a compact C^1-submanifold of X for every $y_0 \varepsilon Y$. In particular, the kernel of every differential $f_*(x): X(x) \to Y(f(x))$ is automatically a direct summand. Application of Theorem 3C yields the

Proposition. *Let $f: X \to Y$ be a C^1-map of Finsler manifolds which foliates X, as in Theorem 3C. If f is proper, then f is a locally C^0-trivial fibration.*

If X is a separable C^k-manifold modeled on a Hilbert space, then McAlpin's extension [4, §4B] of Whitney's imbedding theorem insures that X admits a complete Riemannian structure—and in particular a complete Finsler structure. Furthermore, there is a C^k-partition of unity [8] subordinate to any locally finite atlas on X. Therefore, if $f: X \to Y$ is a proper C^k-foliation, then f is a locally C^k-trivial fibration.

(B) For the next case, suppose that X is a separable C^k-manifold modeled on a Hilbert space. Assume that every differential $f_*(x)$ of the map $f: X \to Y$ is surjective; its kernel $K_x = \operatorname{Ker} f_*(x)$ is a direct summand of $X(x)$. If $K_x^\perp$ denotes its orthogonal complement in $X(x)$, then we require that every $f_*(x) | K_x^\perp \to Y(f(x))$ be an isometry. The splitting s determined by letting

$s(x): Y(f(x)) \to K_x^\perp$ be the inverse of $f_*(x)$ clearly satisfies $\|s(x)\|_x = 1$. An application of Theorem 3C yields the

Proposition. *Let X be a separable complete Riemannian C^k-manifold, and $f: X \to Y$ a surjective C^k-map foliating X. If each restriction $f_*(x) | K_x^\perp \to Y(f(x))$ is an isometry, then f is a locally C^k-trivial fibration.*

That result (in the finite dimensional case) is due to R. Hermann; see [6], [12], and also Rinehart [11] for various refinements and variations.

(C) **Proposition.** *Suppose $f: X \to Y$ satisfies the hypotheses of Theorem 3C. Assume that the following condition is fulfilled:*

For every $y_0 \varepsilon Y$ there is a neighborhood V_0 and a number $\eta_0 > 0$ such that for every $x_0 \varepsilon f^{-1}(V_0)$ there is an $s_{x_0} \varepsilon L(Y(f(x_0)), X(x_0))$ such that $f_*(x_0) \circ s_{x_0}$ is the identity map on $Y(f(x_0))$, and $\|s_{x_0}\| \leq \eta_0$. Then f is a locally C^0-trivial fibration.

Proof. First of all, we take a chart V_0 at y_0 and construct a locally Lipschitz and bounded splitting s on $f^{-1}(V_0)$: For each $x_0 \varepsilon f^{-1}(V_0)$ we extend s_{x_0} to a locally Lipschitz splitting (still called s_{x_0}) in a chart U_{x_0}, such that $\|s_{x_0}(x) - s_{x_0}(x_0)\| \leq 1$; then setting $\eta_0 = \|s_{x_0}(x_0)\|$ we have $\|s_{x_0}(x)\| \leq 1 + \eta_0$. Let (U_α) be a locally finite refinement of the covering $\{ U_{x_0}: x_0 \varepsilon f^{-1}(V_0)\}$, and (λ_α) a locally Lipschitz partition of unity subordinate to (U_α). For each index α choose $x_\alpha \varepsilon U_\alpha \subset U_{x_\alpha}$, and for each $x \varepsilon f^{-1}(V_0)$ define $s(x) = \sum_\alpha \lambda_\alpha(x) s_{x_\alpha}(x)$. Then $f_*(x)s(x) = I$, and $\|s(x)\| \leq \sum_\alpha \lambda_\alpha(x) \|s_{x_\alpha}(x)\| \leq 1 + \eta_0$.

Taking V_0 as a closed neighborhood, we see that $f | f^{-1}(V_0) \to V_0$ is a locally C^0-trivial fibration. But then f itself is, by a well-known general principle (see the Uniformization Theorem in [7]).

Corollary. *Suppose that G is an abstract group which operates on X as a group of C^1-diffeomorphisms and isometries of the Finsler structure. If f is G-invariant ($f(xg) = f(x)$ for all $x \varepsilon X$, $g \varepsilon G$) and G acts transitively on the fibres, then $f: X \to Y$ is a locally C^0-trivial fibration.*

Example. Suppose that G is a group of C^1-isometries operating on the Finsler C^1-manifold X, whose orbits foliate X; see [8, Chapter VI] and [4, §4F]. If $Y = X/G$ is the orbit space and $f: X \to Y$ the orbit map, then the quotient topology on Y is in fact given by the metric

$$\tau(y, y') = \inf \{ \sigma(x, x') : x \varepsilon f^{-1}(y), x' \varepsilon f^{-1}(y') \},$$

where α is the metric of X. Furthermore, Y has induced complete Finsler C^1-structure, and f is a foliation map to which the Corollary applies. We omit the elementary details.

Example. Consider a Fuchsian group Γ operating on the upper half plane U. Let $M(\Gamma)$ be the totality of bounded measurable complex structures on U which are Γ-invariant [1]. $M(\Gamma)$ has a natural complete Finsler structure and a group G of isometries which foliates $M(\Gamma)$. The orbit space (with induced metric) is the Teichmüller space $T(\Gamma)$ of Γ. The Corollary insures that $M(\Gamma)$ is fibred in a locally C^0-trivial manner over $T(\Gamma)$. These ideas and their applications are developed more fully in [3].

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CORNELL UNIVERSITY
CHURCHILL COLLEGE, CAMBRIDGE